MATH 6021 Lecture 7 10/19/2020

Announcements:

1) NO CLASS next week on 10/26
2) Info. about essay / presentation on course webpage

Recall: We introduced "integral currents" to solve the

Oriented Plateau's Problem:

Given $\Gamma^{k-1} \subseteq \mathbb{R}^n$, closed smooth embedded oriented submfd

(treated as a $(k-1)$-dim'l integral cycle)

$\Rightarrow$ $\exists$ k-dim'l integral current T in $\mathbb{R}^n$ which minimizes mass among all k-integral currents T' whose $\partial T' = \Gamma$.

E.g.) Consider two parallel circles as Γ:

Q: How "regular / smooth" is the mass-minimizer $T \subseteq \mathbb{R}^n$?

Interior Regularity (away from Γ) (Ref: L. Simon's book on GMT)

- codim 1 $(k=n-1)$: T is supported on a smooth, oriented hypersurface away from a closed singular set $\mathcal{S}$ with Hausdorff $\dim(\mathcal{S}) \leq n-8$
- higher codim (any k): $\dim(\mathcal{S}) \leq k-2$ Almgren's Big Regularity (>1700 pages)

(optimal: holo. curves in $\mathbb{C}^2$) [c.f. De Lellis et. al.]

Boundary Regularity (along Γ)

- codim 1 : (Hardt-Simon '79) T is smooth along Γ for any n
- higher codim : Still open! (cf. De Lellis et. al.)

Remark: The theory also works in the manifold setting.

Let (M^n, g) be a closed oriented manifold. Fix $\alpha \in H_k(M; \mathbb{Z})$.

Then, $\exists$ homological area-minimizer T, in the sense of current,

with $[T] = \alpha$, and $\dim(\operatorname{sing}(T)) \leq \underbrace{k-7}_{\text{codim } 1}$ or $\underbrace{k-2}_{\text{higher codim}}$.

Some applications of minimal surface theory

Key Idea: Curvature of (M^n, g) $\longleftrightarrow$ stability of min. surfaces

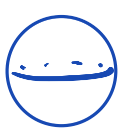
We will be interested in spaces of positive "curvatures":

Sect.	K	$\longrightarrow$ geodesics & their stability
Ricci	Ric	min. surfaces & their stability
Scalar	R	

Q: Does M^n admit a metric of positive sectional / Ric / R ?

Are there any topological obstructions?

Classification of closed orientable surfaces:

   . . . .

Gauss-Bonnet : $K > 0$ $\quad K \equiv 0$ $\quad K < 0$

Thm: Any closed oriented (M^n, g) with $\mathrm{Ric} > 0$ has $H_{n-1}(M;\mathbb{Z}) = 0$.

"**Proof**": Suppose NOT, then $\exists\ 0 \neq \alpha \in H_{n-1}(M;\mathbb{Z})$.

Find a mass-minimizing current T inside the class α.

Know, T is a smooth oriented hypersurface with $\dim(\mathcal{S}) \leq n-8$.

Write $\Sigma := \mathrm{spt}(T)$. Since T is minimizing, Σ is stable.

i.e. $$\int_\Sigma |\nabla\varphi|^2 - (\mathrm{Ric}^M(N,N) + |A|^2)\varphi^2 \geq 0 \qquad \forall \varphi \in C_c^\infty(\Sigma \setminus \mathcal{S})$$

- $\mathcal{S} = \phi$: Take $\varphi \equiv 1$. Contradicts $\mathrm{Ric} > 0$.
- $\mathcal{S} \neq \phi$: use an additional cutoff.

$\square$

Remark: Poincaré duality $\Rightarrow$ $H^1(M;\mathbb{Z}) = 0$ [cf. Lichnerowicz Thm.]

For complete, non-cpt manifolds, we have:

Thm (Schoen-Yau '82)

(M^3, g) complete, non-cpt, $\mathrm{Ric} > 0$ $\Rightarrow$ $M^3 \overset{\text{diffeo.}}{\cong} \mathbb{R}^3$

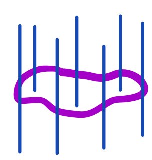
[cf. G. Liu '13 handles the case $\mathrm{Ric} \geq 0$.]

Note: (Sha-Yang ~'90) $n \geq 4$: $\exists$ complicated complete M with $\mathrm{Ric} > 0$.

Remark: For positive sectional curvature [cf. Hopf Conj: $S^2 \times S^2$]

Gromoll-Myer Thm: M^n complete, non-cpt, $K > 0$ $\Rightarrow$ $M^n \overset{\text{diffeo}}{\cong} \mathbb{R}^n$

Cheeger-Gromoll Soul Thm:



S^k

M^n complete, non-cpt, $K \geq 0$ $\Rightarrow$ $M^n \overset{\text{diffeo}}{\cong}$ a vector bundle over a cpt $S^k \subseteq M$ "soul"

Q: Topological obstructions for (M^n, g) with $R_g > 0$?

Geroch Conj: Does T^n admit a metric g with $R_g > 0$?

Approach 1 : index theory for Dirac operator $\longrightarrow$ $\hat{A}$-genus ("spinors")

Approach 2 : minimal surface theory (Schoen-Yau '80)

Schoen-Yau's Trick : (Rewrite stability ineq. st. R^M & R^Σ show up)

Setup: • (M^n, g) closed. $R_g > 0$

• $\Sigma^{n-1} \subseteq M^n$ stable, 2-sided, min. hypersurface. (immersed)

Stability ineq. $\Leftrightarrow$
$$\int_\Sigma \left(Ric^M(e_n, e_n) + |A|^2\right)\varphi^2 \leq \int_\Sigma |\nabla \varphi|^2 \quad \cdots\cdots (*)$$

By Gauss eq^n & minimality,
$$R^\Sigma_{ijij} = R^M_{ijij} + h_{ii} h_{jj} - h_{ij}^2 \qquad \text{where } (h_{ij}) = 2^{nd} \text{ f.f. of } \Sigma$$

$$\xrightarrow[\text{from } 1, \dots, n-1 \text{ (over } \Sigma)]{\text{trace } i,j} \quad R^\Sigma = \sum_{i,j=1}^{n-1} R^\Sigma_{ijij} = \sum_{i,j=1}^{n-1} R^M_{ijij} + 0 - |A|^2$$

Note: $$Ric^M(e_n, e_n) = \sum_{i=1}^{n-1} R^M_{inin} = \frac{1}{2}\left(\underbrace{\sum_{i,j=1}^{n} R^M_{ijij}}_{= R^M} - \sum_{i,j=1}^{n-1} R^M_{ijij} \right)$$

i.e. $$Ric^M(e_n, e_n) = \frac{1}{2} R^M - \frac{1}{2} R^\Sigma - \frac{1}{2} |A|^2 .$$

Plug into $(*)$.

$$(**) \cdots\cdots \frac{1}{2} \int_\Sigma \left(R^M - R^\Sigma + |A|^2\right) \varphi^2 \leq \int_\Sigma |\nabla \varphi|^2 \quad \forall \varphi \in C_c^\infty(\Sigma)$$

Schoen-Yau's Dimension Reduction Thm:

Let (M^n, g) be a closed manifold with $R^M > 0$, $n \geq 4$.

Then, any 2-sided stable min. hypersurface $\Sigma^{n-1} \subseteq M^n$ is "Yamabe positive", i.e. $\exists$ metric $\tilde{g}$ on Σ which is conformally equivalent to $g|_\Sigma$ s.t $\tilde{g}$ has $R^\Sigma_{\tilde{g}} > 0$.

"Sketch of Proof":

Setup: $g = g|_\Sigma$ induced metric on $\Sigma \subseteq (M^n, g)$

$\tilde{g} := u^{\frac{4}{n-3}} g$ conformal metric on Σ, where $0 < u \in C^\infty(\Sigma)$.

- The scalar curvatures on Σ w.r.t g & $\tilde{g}$ are related by

$$R^\Sigma_{\tilde{g}} = -c(n)^{-1} u^{-\frac{n+1}{n-3}} (L_0 u)$$

where $L_0 u := \Delta_g u - c(n) R^\Sigma_g u$ "conformal Laplacian" of (Σ, g)

Here: $c(n) := \frac{n-3}{4(n-2)} < \frac{1}{4}$.

- $(**)$ & $R^M > 0$ then implies

$$\int_\Sigma \left(-\Delta_g \varphi + \frac{1}{2} R^\Sigma_g \varphi\right) \varphi > 0 \qquad \forall\, 0 \neq \varphi \in C^\infty(\Sigma)$$

$$\Rightarrow \int_\Sigma \left(-\underbrace{2c(n)}_{<1} \Delta_g \varphi + c(n) R^\Sigma_g \varphi\right) \varphi > 0 \qquad \forall\, 0 \neq \varphi \in C^\infty(\Sigma)$$

i.e $\lambda_1(-L_0) > 0$. Take $0 < u \in C^\infty(\Sigma)$ be the 1^{st} eigenfcn of $-L_0$

Then, $R^\Sigma_{\tilde{g}} = c(n)^{-1} u^{-\frac{n+1}{n-3}} \underbrace{(-L_0 u)}_{= \lambda_1(-L_0) u} > 0$ everywhere.

$\blacksquare$

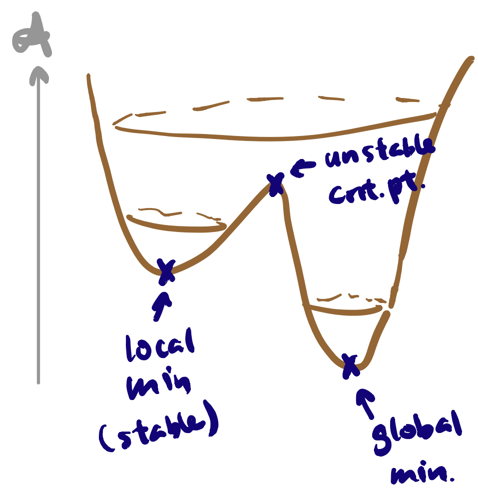
Remark: When $n=3$, same argument $\Rightarrow \Sigma^2 \approx S^2$. [eg $M^3 = S^1 \times S^2$]

"Cor": Geroch Conjecture is true for any n.

[Schoen-Yau '79 ($n \leq 7$) ; Schoen-Yau '17 (any n).]

The beginning of min-max theory

Recall: min. surface $\longleftrightarrow$ crit. points to the area functional $\mathcal{A}$



$\mathcal{S} = \{$"surfaces"$\}$

∞-dimensional

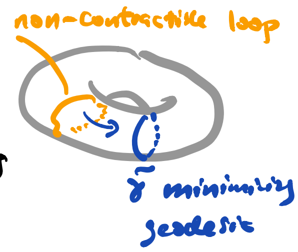
Q: Is there a "Morse theory" for $\mathcal{A}$?

Sub-Q: How to construct "unstable" min. surfaces ?

A: min-max theory !

In 1D case, closed min surf. are just closed geodesics.

Question: (Poincaré) Every closed (M^2, g) admits a closed geodesic ?

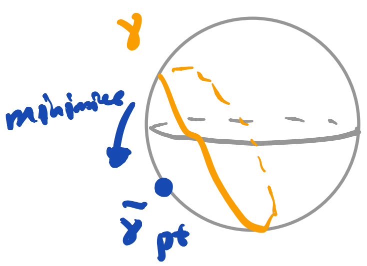


- Easy case: genus$(M^2) > 0$

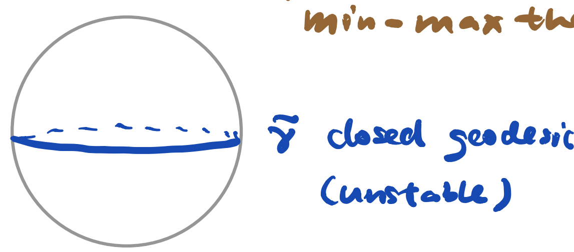
  minimize length in a non-trivial free homotopy class

- Difficult case: genus$(M^2) = 0$, ie $M^2 \approx S^2$.

Birkhoff (~1920's): Every (S^2, g) admits a closed geodesic.



"min-max theory"

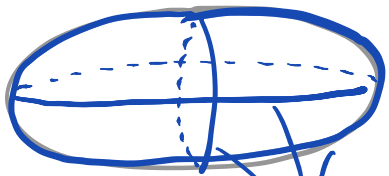


Q: What about more than one geodesics?

Lusternik-Schnirelmann '47 / Grayson '89:

Every (S^2, g) admits 3 simple closed geodesics.

Remark: This is optimal, e.g. by ellipsoids



3 simple closed geodesics.

Q: What about including immersed geodesics?

Franks '92, Bangert '93: Every (S^2, g) admits ∞'ly many closed geodesics that are "geometrically distinct".

Q: What about min. hypersurfaces in (M^n, g)?

A: Almgren-Pitts '81:

Every (M^n, g), $3 \le n \le 6$, admit ONE smooth closed embedded min. hypersurface Σ^{n-1}.

Schoen-Simon '81:

Same holds for n, except that there is possibly a singular set $\mathcal{S} \subseteq \Sigma^{n-1}$ s.t. $\dim(\mathcal{S}) \le n-8$.

Yau's Conjecture (1982 Problem Sections)

Every closed (M^3, g) admits ∞'ly many smooth closed embedded min. surfaces.

- Recently solved Marques-Neves '17, A. Song '18